Bivariate Krawtchouk polynomials: Inversion and connection problems with the NAVIMA algorithm

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In memory of Pablo González Vera

MSC:
primary 33C65
secondary 33C50
39A14

Keywords:
Bivariate discrete orthogonal polynomials
Bivariate Krawtchouk polynomials
Inversion problems
Connection problems
Partial difference equations
Generalized Appell hypergeometric series

A B S T R A C T

In this paper we present a recurrent procedure to solve an inversion problem for monic bivariate Krawtchouk polynomials written in vector column form, giving its solution explicitly. As a by-product, a general connection problem between two vector column of monic bivariate Krawtchouk families is also explicitly solved. Moreover, in the non monic case and also for Krawtchouk families, several expansion formulas are given, but for polynomials written in scalar form.

1. Introduction

Let $\{P_n(x)\}_n$ and $\{Q_n(x)\}_n$ be two polynomial families in one variable such that $P_n(x)$ and $Q_m(x)$ are of degree exactly n and m, respectively. The connection problem between these two families asks for the computation of the coefficients $D_m(n)$ in the expansion

$$P_n(x) = \sum_{m=0}^{n} D_m(n) Q_m(x). \tag{1}$$

To deal with the connection problem (1) there exist a number of methods in the literature (see for instance [1–8] and the references therein). In particular, it can be recurrently solved using an algorithm (the *Navima* algorithm) developed

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and successfully applied by the authors in several univariate situations (see e.g. [1,3,9,7,10]). This algorithm generates in a systematic way a linear recurrence relation in m for $D_m(n)$ which, sometimes, can be solved explicitly.

A different approach, based on the so-called "inversion problems", was adopted in [11,12]. Key point in this strategy is to find an appropriate basis $\{\vartheta_m(x)\}_m$ allowing an explicit expansion as

$$P_n(x) = \sum_{m=0}^{n} A_m(n)\vartheta_m(x). \tag{2}$$

Then, knowledge of the $A_m(n)$-coefficients together with the solution of the "inversion problem"

$$\vartheta_n(x) = \sum_{m=0}^{n} C_m(n)Q_m(x), \tag{3}$$

(using e.g. the *Navima* algorithm as done in [11]) gives the searched connection coefficients as a nested sum of the form

$$D_m(n) = \sum_{j=m}^{n} A_j(n)C_m(j). \tag{4}$$

The main aim of this paper is to extend these ideas in order to solve some inversion and connection problems for monic Krawtchouk polynomials in two discrete variables.

The use of quantities that are determined on a discrete set of argument values is, nowadays, widely extended in a great variety of fields ranging from optical image processing, to probability and coding theory, and also in quantum information theory, and finite oscillator models, among others. As two outstanding examples of this discrete situation we can mention the Clebsch–Gordan and Racah coefficients (the $3j$- and $6j$-symbols) and the classical orthogonal polynomials of a discrete variable (the Hahn, Meixner, Krawtchouk and Charlier polynomials). It is well known that, in particular, both $3j$- and $6j$-symbols of the oscillator algebra can be expressed in terms of the Krawtchouk polynomials in one discrete variable [13].

In this paper we shall deal with monic bivariate Krawtchouk polynomials. There exist different extensions of univariate Krawtchouk polynomials [14] to bivariate and multivariate contexts [15–25]. Here we consider those introduced in Section 2 by means of a generalized Appell hypergeometric series [24, Eq. (3.4)].

In Section 3 a natural inversion problem for these monic bivariate Krawtchouk family written in vector form is solved by using an extension of the *Navima* algorithm to this bivariate case. Then, using this special inversion problem, an explicit formula is obtained for the connection coefficients between two families of monic bivariate Krawtchouk polynomials (in vector form) with different parameters. Moreover, the non monic case is considered giving some specific connection formulas between non-monic and monic bivariate Krawtchouk families but written in scalar form.

2. Monic bivariate Krawtchouk polynomials

Bivariate monic Krawtchouk polynomials of total degree $n_1 + n_2$, are defined by Tratnik [24, Eq. (3.4)] in terms of the generalized Appell hypergeometric series [26] for $0 \le n_1 + n_2 \le N$,

$$\begin{aligned}\widehat{K}^{p_1,p_2}_{n_1,n_2}(x, y; N) &= p_1^{n_1} p_2^{n_2} (-N)_{n_1+n_2}\, F^{0:2;2}_{1:0;0}\left(\begin{array}{c} - : -n_1, -x; -n_2, -y \\ -N : -; - \end{array} \middle| \, \frac{1}{p_1}, \frac{1}{p_2} \right), \\ &= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \frac{p_1^{n_1-i}\, p_2^{n_2-j}\, (-n_1)_i\, (-x)_i\, (-n_2)_j\, (-y)_j\, (-N)_{n_1+n_2}}{i!\, j!\, (-N)_{i+j}},\end{aligned} \tag{5}$$

where $(A)_s = A(A+1)\cdots(A+s-1)$ denotes the Pochhammer symbol, N is a non-negative integer and p_1, p_2 are real parameters such that:

$$p_1 > 0, \qquad p_2 > 0, \qquad 0 < p_1 + p_2 < 1.$$

These polynomials satisfy the following orthogonality relation

$$\sum_{x=0}^{N} \sum_{y=0}^{N-x} \widehat{K}^{p_1,p_2}_{m_1,m_2}(x, y; N)\, \widehat{K}^{p_1,p_2}_{s_1,s_2}(x, y; N)\, \varrho(x, y; p_1, p_2, N) = \Lambda_S(p_1, p_2; N)\delta_{M,S}, \tag{6}$$

where $S = s_1 + s_2,\ M = m_1 + m_2$ with $S, M \le N$ and

$$\begin{aligned}\Lambda_S(p_1, p_2; N) &= (-1)^{N+S}\, (N-S+1)_S\, p_1^{s_1}\, p_2^{s_2}\, (p_1+p_2-1)^{-N}\, s_1! s_2! \\ &\quad \times (p_1-1)^{s_1}\, (p_2-1)^{s_2}\, {}_2F_1\left(\begin{array}{c} -s_2, -s_1 \\ 1 \end{array} \middle| \, \frac{p_1 p_2}{(p_1-1)(p_2-1)} \right).\end{aligned}$$

Up to a normalizing constant factor, the corresponding weight function is given by

$$\varrho(x, y; p_1, p_2, N) = \frac{{p_1}^x\, (1-p_1-p_2)^{-x-y}\, {p_2}^y\, N!}{x!\, (N-x-y)!\, y!}, \tag{7}$$

which coincides with the trinomial distribution [18,24,27]. Limit relation from trinomial distribution to bivariate Gaussian distribution has been analyzed in [28], giving rise to the limit relation between bivariate Krawtchouk polynomials and bivariate Hermite polynomials introduced by Hermite [26, p. 373].

We consider here the column vector representation of these monic orthogonal polynomials defined as

$$\begin{aligned}\mathbb{K}_n(x,y) &= \mathbb{K}_n^{p_1,p_2}(x,y;N)\\ &= (\widehat{K}_{n,0}^{p_1,p_2}(x,y;N),\ldots,\widehat{K}_{n-i,i}^{p_1,p_2}(x,y;N),\ldots,\widehat{K}_{0,n}^{p_1,p_2}(x,y;N))^{\mathrm{T}}.\end{aligned} \tag{8}$$

Among other algebraic and difference properties [29], these monic bivariate Krawtchouk polynomials, $\mathbb{K}_n(x,y)$ (column vector of dimension $n+1$), satisfy the following structure relation [29] we use later on:

$$x\nabla_1\mathbb{K}_n(x,y)+y\nabla_2\mathbb{K}_n(x,y)=\mathbb{S}_n\mathbb{K}_n(x,y)+\mathbb{T}_n\mathbb{K}_{n-1}(x,y),\quad n\ge 1, \tag{9}$$

where

$$\begin{aligned}\nabla_1\mathbb{K}_n(x,y)&=\mathbb{K}_n(x,y)-\mathbb{K}_n(x-1,y),\\ \nabla_2\mathbb{K}_n(x,y)&=\mathbb{K}_n(x,y)-\mathbb{K}_n(x,y-1),\end{aligned}$$

the matrix $\mathbb{S}_n=n\mathbb{I}_{n+1}$ of size $(n+1)\times(n+1)$ is diagonal and for $n\ge 1$ the matrix $\mathbb{T}_n$ of size $(n+1)\times n$ is a lower bi-diagonal matrix given by

$$\mathbb{T}_n=\mathbb{T}_n(p_1,p_2,N)=\begin{pmatrix} t_{0,0} & & & & \bigcirc\\ t_{1,0} & t_{1,1} & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & t_{n-1,n-2} & t_{n-1,n-1}\\ \bigcirc & & & 0 & t_{n,n-1}\end{pmatrix}, \tag{10}$$

with

$$t_{i,i}=(n-i)(1-n+N)p_1,\qquad t_{i+1,i}=(i+1)(1-n+N)p_2\quad(0\le i\le n-1).$$

3. Inversion and connection problems and the bivariate Navima algorithm

When trying to extend to the bivariate situation the ideas described in the introduction for the one variable case, first step is to find a natural basis $\{\mathbb{X}^n\}_n$, playing the role of the already mentioned $\{\vartheta_m(x)\}_m$ (see Eq. (2)) in the univariate case, in which to expand the vector form of monic Krawtchouk polynomials (8) as:

$$\mathbb{K}_n^{q_1,q_2}(x,y;M)=\sum_{j=0}^{n}\mathbb{A}_j(n)\mathbb{X}^j,\quad n\le M, \tag{11}$$

where $\mathbb{A}_j(n)$ is a $(n+1)\times(j+1)$ matrix, and $\mathbb{X}^j$ is a $(j+1)\times 1$ vector.

The hypergeometric representation (5) suggests to use as $\{\mathbb{X}^n\}_n$-basis the monomials in vector form

$$\mathbb{X}^n=((-x)_n(-y)_0,(-x)_{n-1}(-y)_1,\ldots,(-x)_1(-y)_{n-1},(-x)_0(-y)_n)^T. \tag{12}$$

In fact, this basis turns out to be very appropriate because the lower triangular matrices $\mathbb{A}_j(n)$ can be directly obtained from the hypergeometric representation (5). It is as follows: $\mathbb{A}_n(n)=(-1)^n\mathbb{I}_{n+1}$, where $\mathbb{I}_{n+1}$ is the identity matrix of size $n+1$, and for $0\le j\le n-1$ the coefficients of each matrix $\mathbb{A}_j(n)$ are given by

$$(\mathbb{A}_j(n))_{r,s}=(-1)^n\begin{cases}0, & r+1\le s\quad\text{or } s+n-j+1\le r,\\ a_{r,s}(j,n,q_1,q_2,M), & \text{otherwise},\end{cases} \tag{13}$$

being

$$a_{r,s}(j,n,q_1,q_2,M)=\frac{(r-1)!(n-r+1)!q_2^{r-s}(M-n+1)_{n-j}\binom{n-j}{r-s}q_1^{s-j+n-r}}{(s-1)!(j-s+1)!(n-j)!},$$

for $r=1,\ldots,n+1$ and $s=1,\ldots,j+1$.

Next step is to consider the following bivariate inversion problem

$$\mathbb{X}^n=\sum_{m=0}^{n}\mathbb{C}_m(n)\mathbb{K}_m^{p_1,p_2}(x,y;N),\quad n\le N, \tag{14}$$

which is the bivariate analogous to Eq. (3). So, our problem now is to compute the connection coefficients $\mathbb{C}_m(n) = \mathbb{C}_m(n; p_1, p_2, N)$ (which are matrices of size $(n+1) \times (m+1)$) giving the expansion of the $\mathbb{X}^n$ vector basis in terms of column vector monic Krawtchouk polynomials $\mathbb{K}_m(x, y) = \mathbb{K}_m^{p_1,p_2}(x, y; N)$ defined in (8). The coefficients $\mathbb{C}_m(n)$ in (14) have been already obtained in [30, Theorem 4.4] by complete different means.

To solve (14) in the vector form here considered, we first realize that monic normalization of $\mathbb{K}_m(x, y)$ and Eqs. (12)–(14) give:

$$\mathbb{C}_n(n) = (-1)^n \mathbb{I}_{n+1}. \tag{15}$$

For the computation of the remaining matrix connection coefficients $\mathbb{C}_m(n)$ $(0 \le m \le n-1)$, we are going to use the bivariate extension of the *Navima* algorithm. So, the inversion problem (14) serves us to illustrate how it works in this two discrete variables situation.

The working hypothesis of the algorithm [1,3,7] requires the existence of a difference operator killing the left hand side of Eq. (14). We have, for $0 \le j \le n$,

$$x\nabla_1 \left[(-x)_{n-j}(-y)_j\right] + y\nabla_2 \left[(-x)_{n-j}(-y)_j\right] - n \left[(-x)_{n-j}(-y)_j\right] = 0,$$

and as a consequence,

$$x\nabla_1 \mathbb{X}^n + y\nabla_2 \mathbb{X}^n - n\mathbb{X}^n = 0. \tag{16}$$

So, we can now follow the steps prescribed by the algorithm applying the operator (16) to both sides of (14) and use (9) as well as $\mathbb{K}_{-1}(x, y) = 0$ to obtain the expression:

$$\begin{aligned} 0 &= \sum_{m=0}^{n} \mathbb{C}_m(n) \left\{\mathbb{S}_m \mathbb{K}_m(x, y) + \mathbb{T}_m \mathbb{K}_{m-1}(x, y) - n\mathbb{K}_m(x, y)\right\} \\ &= \sum_{m=0}^{n} \mathbb{C}_m(n) \left\{(\mathbb{S}_m - n\mathbb{I}_{m+1})\, \mathbb{K}_m(x, y) + \mathbb{T}_m \mathbb{K}_{m-1}(x, y)\right\}. \end{aligned}$$

A shift of indices gives:

$$\begin{aligned} 0 &= \sum_{m=0}^{n} \left\{\mathbb{C}_m(n)\, (\mathbb{S}_m - n\mathbb{I}_{m+1}) + \mathbb{C}_{m+1}(n)\mathbb{T}_{m+1}\right\} \mathbb{K}_m(x, y) \\ &= \sum_{m=0}^{n} \left\{(m-n)\mathbb{C}_m(n) + \mathbb{C}_{m+1}(n)\mathbb{T}_{m+1}\right\} \mathbb{K}_m(x, y), \end{aligned}$$

with the initial condition $\mathbb{C}_{n+1}(n) = 0$. Finally, the latter expression together with the linear independence of the vector column family $\{\mathbb{K}_m(x, y)\}_m$, provides the usual output of the *Navima* algorithm which is, in this case, the following backward recurrence formula for the inversion coefficients:

$$(m-n)\mathbb{C}_m(n) + \mathbb{C}_{m+1}(n)\mathbb{T}_{m+1} = 0 \Leftrightarrow \mathbb{C}_m(n) = \frac{1}{n-m}\, \mathbb{C}_{m+1}(n)\, \mathbb{T}_{m+1}, \quad (0 \le m \le n-1), \tag{17}$$

with the already obtained initial condition (15).

In the particular case we have considered, it turns out that this backward recurrence (17) can be explicitly solved in terms of the matrix $\mathbb{T}_n = \mathbb{T}_n(p_1, p_2, N)$ defined in (10). The solution of (14) is given by:

$$\mathbb{C}_m(n) \equiv \mathbb{C}_m(n; p_1, p_2, N) = \frac{(-1)^n}{(n-m)!} \prod_{j=0}^{n-m-1} \mathbb{T}_{n-j}. \tag{18}$$

By using the explicit expression of the matrices $\mathbb{T}_n$ given in (10), we obtain that the coefficients of the matrices $\mathbb{C}_m(n; p_1, p_2, N)$ can be explicitly obtained as

$$(\mathbb{C}_m(n; p_1, p_2, N))_{i,j} = (-1)^n \begin{cases} 0, & i+1 \le j \quad \text{or}\, j+n-m+1 \le i, \\ c_{i,j}(m, n, p_1, p_2, N), & \text{otherwise}, \end{cases} \tag{19}$$

where

$$c_{i,j}(m, n, p_1, p_2, N) = \frac{(i-1)!(n-i+1)!\,p_2^{i-j}(N-n+1)_{n-m} \binom{n-m}{i-j} p_1^{j-m+n-i}}{(j-1)!(m-j+1)!(n-m)!},$$

for $i = 1, \dots, n+1$ and $j = 1, \dots, m+1$.

Finally, it is worth mentioning that, for the monic Krawtchouk polynomials (8) we have considered here, the connection coefficients (18) of the inversion problem (14) and those (13) of the corresponding direct problem (11) exactly coincide, i.e.:

$$\mathbb{A}_j(n) = \mathbb{C}_j(n; q_1, q_2, M), \quad 0 \le j \le n \le M.$$

3.1. Solving connection via inversion

Having at hand the solutions of the direct (11) and inverse (14) problems, it is possible to finish with the discrete bivariate extension of the strategy for solving connection problems as nested sums by means of inversion formulas, already used by the authors in the one variable situation [11,12].

Let us consider, as illustration, the connection problem between two column vector monic Krawtchouk families (8) with different parameters; i.e.:

$$\mathbb{K}_n^{q_1,q_2}(x, y; M) = \sum_{m=0}^{n} \mathbb{D}_m(n) \mathbb{K}_m^{p_1,p_2}(x, y; N), \tag{20}$$

which make sense if $0 \le m \le n \le M \le N$, being N, M non-negative integers and $p_1,\ p_2,\ q_1,\ q_2$ real parameters satisfying

$$p_1, q_1 > 0, \qquad p_2, q_2 > 0, \qquad 0 < p_1 + p_2 < 1, \qquad 0 < q_1 + q_2 < 1.$$

In order to compute the connection coefficients $\mathbb{D}_m(n)$ of this expansion, we use the inverse problem (14) to replace $\mathbb{X}^j$ in the direct problem expression (11) to obtain:

$$\begin{aligned} \mathbb{K}_n^{q_1,q_2}(x, y; M) &= \sum_{j=0}^{n} \mathbb{A}_j(n) \left(\sum_{m=0}^{j} \mathbb{C}_m(j) \mathbb{K}_m^{p_1,p_2}(x, y; N) \right) \\ &= \sum_{m=0}^{n} \left(\sum_{j=m}^{n} \mathbb{A}_j(n) \mathbb{C}_m(j) \right) \mathbb{K}_m^{p_1,p_2}(x, y; N). \end{aligned} \tag{21}$$

An appropriate rearrange of the latter nested sum leads to the searched solution which, for $0 \le m \le n-1$, is:

$$\mathbb{D}_m(n) = \sum_{j=m}^{n} \mathbb{A}_j(n) \mathbb{C}_m(j) = \sum_{j=m}^{n} \mathbb{C}_j(n; q_1, q_2, M) \mathbb{C}_m(j; p_1, p_2, N), \tag{22}$$

where $\mathbb{A}_j(n)$ is given in (13) and $\mathbb{D}_n(n) = \mathbb{I}_{n+1}$.

In this way, the connection problem (20) is solved via the inversion formula (14) which in turn we have solved by means of the bivariate extension of the *Navima* algorithm.

In the $N = M$ particular case, the sum (22) gets simplified as follows:

$$\sum_{j=m}^{n} \mathbb{C}_j(n; q_1, q_2, N) \mathbb{C}_m(j; p_1, p_2, N) = (-1)^n \, \mathbb{C}_m(n; p_1 - q_1, p_2 - q_2, N), \tag{23}$$

for $0 \le m \le n \le N$, where the above relation is just a consequence of (19) and the iterated use of the binomial theorem for each element of the left hand side matrix.

Therefore, the connection coefficients in the problem (20) are now explicitly given by

$$\mathbb{D}_m(n) = (-1)^n \mathbb{C}_m(n; p_1 - q_1, p_2 - q_2, N), \quad 0 \le m \le n \le N,$$

where the matrices $\mathbb{C}_m(n; p_1, p_2, N)$ are given in Eqs. (18) and (19). Notice that these matrices solve the inversion problem (14) for $p_1, p_2 > 0$, although these matrices are defined for any value of the parameters as shown by their explicit expressions (19).

3.2. Connection formulas linking different families of bivariate Krawtchouk polynomials

Bivariate monic Krawtchouk polynomials defined in (5) are solution of the following second-order partial difference equation of hypergeometric type [31–34]

$$\begin{aligned} &(p_1 - 1)x\Delta_1\nabla_1 u(x, y) + p_1 y \Delta_1 \nabla_2 u(x, y) + p_2 x \Delta_2 \nabla_1 u(x, y) + (p_2 - 1) y \Delta_2 \nabla_2 u(x, y) \\ &\quad + (x - Np_1)\Delta_1 u(x, y) + (y - Np_2)\Delta_2 u(x, y) - (n_1 + n_2) u(x, y) = 0, \end{aligned} \tag{24}$$

where the forward and backward difference operators are defined by

$$\begin{aligned} \Delta_1 u(x, y) &= u(x+1, y) - u(x, y), \qquad & \Delta_2 u(x, y) &= u(x, y+1) - u(x, y), \\ \nabla_1 u(x, y) &= u(x, y) - u(x-1, y), \qquad & \nabla_2 u(x, y) &= u(x, y) - u(x, y-1). \end{aligned}$$

This equation has at least two more orthogonal polynomial solutions both non monic and introduced by Tratnik [32], namely:

- the non-monic bivariate Krawtchouk polynomials defined as the generalized Appell hypergeometric series [26]

$$\begin{aligned} K_{n_1,n_2}^{p_1,p_2}(x,y;N) &= (x+y-N)_{n_1+n_2}\, F^{0:2;2}_{1:0;0}\left(\begin{array}{c} -:-n_1,-x;-n_2,-y \\ -n_1-n_2-x-y+N+1:-;- \end{array} \,\middle|\, \frac{\gamma}{p_1},\frac{\gamma}{p_2} \right) \\ &= \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \frac{(x+y-N)_{n_1+n_2}\,(-n_1)_i\,(-x)_i\,(-n_2)_j\,(-y)_j}{(-n_1-n_2-x-y+N+1)_{i+j}\,i!\,j!}\left(\frac{\gamma}{p_1}\right)^i\left(\frac{\gamma}{p_2}\right)^j \end{aligned} \tag{25}$$

  where $\gamma = p_1+p_2-1$, for $0 \le n_1+n_2 \le N$, and,
- the non-monic families defined for $0 \le n_1+n_2 \le N$ as a product of two univariate Krawtchouk polynomials

$$\widetilde{K}_{n_1,n_2}^{p_1,p_2}(x,y;N) = \frac{(N-n_1)!}{N!\,(n_1-N)_{n_2}} K_{n_1}(x;p_1/(p_1+p_2),x+y)\,K_{n_2}(x+y-n_1;p_1+p_2,N-n_1), \tag{26}$$

  where the univariate Krawtchouk polynomials are given by

$$K_n(x;p,N) = (-N)_n\,{}_2F_1\left(\begin{array}{c} -n,-x \\ -N \end{array} \,\middle|\, \frac{1}{p} \right), \quad 0<p<1,\ 0\le n\le N.$$

These families (25) and (26) are also orthogonal with respect to (7) on the same domain G defined by $x \ge 0$, $y \ge 0$, and $0 \le x+y \le N$.

Then, it is interesting to consider the two connection problems between the vector forms of these two non-monic polynomials and the monic family (5). However, it is very difficult to solve them or even to give a reasonable recurrence for their corresponding matrix coefficients. What is possible in this case is to derive some relations between these two families and the monic one, linking scalar polynomials of the same total degree as it is done in [30, Proposition 4.2] by very different means than the ones we use here. Just as illustration of the type of relations we are talking about, we give here the following examples for specific elements of these families. These results are obtained using the definition of the Krawtchouk families given in (5), (25), and (26), and equating the coefficients in $x^r y^s$.

- Expansions of the non monic family (25) in the monic one (5):

$$\begin{aligned} K_{n,0}^{p_1,p_2}(x,y;N) &= \sum_{j=0}^{n} \frac{(1-p_2)^{n-j}p_1^j}{(n-j)!j!}\widehat{K}_{n-j,j}^{p_1,p_2}(x,y;N), \\ K_{0,n}^{p_1,p_2}(x,y;N) &= \sum_{j=0}^{n} \frac{(1-p_1)^{n-j}p_2^j}{(n-j)!j!}\widehat{K}_{n-j,j}^{p_1,p_2}(x,y;N), \\ K_{n-1,1}^{p_1,p_2}(x,y;N) &= \sum_{j=0}^{n} \frac{(1-p_2)^{n-j-1}p_1^{j-1}}{(n-j)!j!}\,(j(1-p_1)(1-p_2)+(n-j)p_1p_2)\,\widehat{K}_{n-j,j}^{p_1,p_2}(x,y;N), \\ K_{1,n-1}^{p_1,p_2}(x,y;N) &= \sum_{j=0}^{n} \frac{(1-p_1)^{n-j-1}p_2^{j-1}}{(n-j)!j!}\,((n-j)(1-p_1)(1-p_2)+jp_1p_2)\,\widehat{K}_{n-j,j}^{p_1,p_2}(x,y;N). \end{aligned}$$

- Expansions of the non monic family (26) in the monic one (5):

$$\begin{aligned} \widetilde{K}_{n,0}^{p_1,p_2}(x,y;N) &= K_n\left(x;\frac{p_1}{p_1+p_2},x+y\right) \\ &= (-x-y)_n\,{}_2F_1\left(\begin{array}{c} -n,-x \\ -x-y \end{array} \,\middle|\, \frac{p_1+p_2}{p_1} \right) \\ &= \sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(\frac{p_2}{p_1}\right)^{n-j}\widehat{K}_{n-j,j}^{p_1,p_2}(x,y;N), \\ \widetilde{K}_{0,n}^{p_1,p_2}(x,y;N) &= \frac{1}{(-N)_n}K_n\,(x+y;p_1+p_2,N) \\ &= {}_2F_1\left(\begin{array}{c} -n,-x-y \\ -N \end{array} \,\middle|\, \frac{1}{p_1+p_2} \right) \\ &= \frac{(-1)^n}{(p_1+p_2)^n(N-n+1)_n}\sum_{j=0}^{n}\binom{n}{j}\widehat{K}_{n-j,j}^{p_1,p_2}(x,y;N). \end{aligned}$$

Moreover,

$$\widetilde{K}_{n-1,1}^{p_1,p_2}(x,y;N) = \frac{p_1^{1-n}}{(p_1+p_2)(N-n+1)_n} \times \sum_{j=0}^{n}(-1)^j\left[\binom{n-1}{j-1}p_1^{j-1}p_2^{n-j} - \binom{n-1}{j}p_1^j p_2^{n-j-1}\right]\widehat{K}_{n-j,j}^{p_1,p_2}(x,y;N),$$

and

$$\widetilde{K}_{1,n-1}^{p_1,p_2}(x,y;N) = \frac{(-1)^{n-1}}{p_1(p_1+p_2)^{n-1}(N-n+1)_n} \times \sum_{j=0}^{n}\left[\binom{n-1}{j}p_2 - \binom{n-1}{j-1}p_1\right]\widehat{K}_{n-j,j}^{p_1,p_2}(x,y;N).$$

Constructive remark

Other sophisticated situations can also be investigated including, for instance, Meixner or Hahn bivariate polynomials. More structural and difference properties satisfied by these families (see [29]) must there be used allowing to build recurrence relations for the connection coefficients as in Eq. (17), but involving three or more terms, and therefore difficult to solve in the general matrix case.

Acknowledgments

The authors thank the anonymous reviewers for several suggestions that improved the paper significantly. This work was partially supported by Ministerio de Economía y Competitividad of Spain under grants MTM2012–38794–C02–01 and MTM2011–28952–C02, cofinanced by the European Community fund FEDER. A. Ronveaux also thanks the Departamento de Matemática Aplicada II of Universidade de Vigo for the kind invitations and financial support. AZ also acknowledges partial support from Universidad Politécnica de Madrid, Spain ("TACA" consolidated research group) and from Junta de Andalucía (Spain) under contracts P09-TEP-5022 and P11-FQM-7276.

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